FLOW GENERATED IN THE AMBIENT MEDIUM BY A TURBULENT JET

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UDC 532.517.4

A solution is obtained to the problem of calculating the flow induced in the ambient medium by a plane jet expelled from a nozzle of finite dimensions. The theoretical results are in satisfactory agreement with the experiment.

In a number of technical applications it is of interest to study the influx of a fluid toward a jet caused by the ejecting action of the jet. A solution to the problem of the fluid flow induced by a source-jet is given in [1], while for the flow induced by plane and fan-shaped jets expelled from infinitely narrow slots solutions are given in [2].

An attempt to take the initial portion of the jet into account has been made in [3], where the pressure distribution at the wall was determined by the source-sink method for a jet expelled from an unbounded wall normal to the axis of the jet.

Figure 1 shows the flow pattern for a plane turbulent submerged jet expelled from a nozzle with a width $2b_0$. In accordance with [4], the jet boundaries are assumed to be rectilinear, while the transverse velocity component v_θ at the initial (for $r \le r_0$) and main (for $r \ge r_0$) portions of the jet are defined, respectively, by the relations

$$v_\theta = -0.036\, U_a; \quad v_\theta = -\frac{0.181\sqrt{b_0}\, U_a}{\sqrt{r}}. \tag{1}$$

The fluid flowing to the jet is assumed to be ideal and incompressible, and the flow to be a potential one. In this statement, the calculation of the velocity field beyond the jet reduces to the solution of the Laplace equation for the stream function

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0 \tag{2}$$

with the boundary conditions

$$\text{for } r = 0 \quad \psi = 0, \quad \text{for } r \to \infty \quad \frac{\partial \psi}{\partial r} \to 0, \quad \frac{1}{r}\frac{\partial \psi}{\partial \theta} \to 0; \quad \text{for } \theta = \alpha \quad \psi = 0, \quad \text{for } \theta = 0 \quad \psi = f(r). \tag{3}$$

The value of the stream function at the jet boundary is calculated from the known value of the transverse velocity component (1)

$$\psi(r, 0) = \int v_\theta\, dr,$$

from where

$$f(r) = \begin{cases} k_1 r & \text{for } 0 \leqslant r \leqslant r_0, \\ k_1 r_0 + k_2(\sqrt{r} - \sqrt{r_0}) & \text{for } r_0 \leqslant r < \infty, \end{cases} \tag{4}$$

where $k_1 = -0.036\, U_\alpha$, and $k_2 = -0.362\sqrt{b_0}U_\alpha$.

By introducing a new required function $W(r, \theta)$

$$W(r, \theta) = \psi(r, \theta) + f(r)\left(\frac{\theta}{\alpha} - 1\right) \tag{5}$$

Mechanics Institute, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 15, No. 5, pp. 836-842, November, 1968. Original article submitted November 14, 1966.

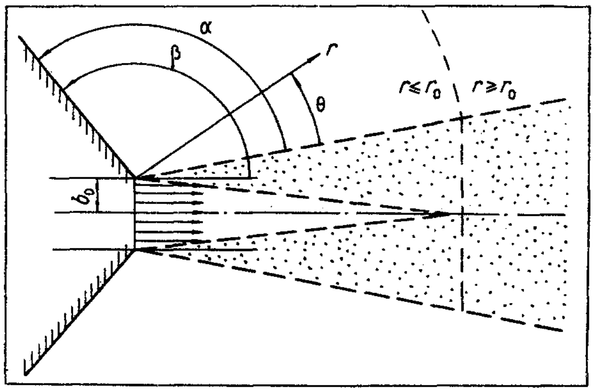

Fig. 1

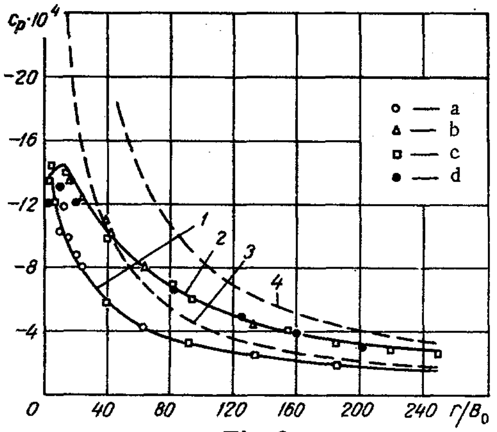

Fig. 2

Fig. 1. Flow pattern.

Fig. 2. Pressure distribution over the nozzle wall ($\theta = \alpha$). Calculated from (17)-(19): 1) $\beta = 3\pi/4$; 2) $\beta = \pi/2$; calculated from (21)-(23): 3) $\beta = 3\pi/4$; 4) $\beta = \pi/2$; experiment: a) $U_\alpha = 80$ m/sec; b) 100 m/sec; c 150 m/sec; d) experimental data [3].

Eq. (2) reduces to an inhomogeneous differential equation

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = \left(f'' + \frac{1}{r} f'\right)\left(\frac{\theta}{\alpha} - 1\right) \tag{6}$$

with homogeneous boundary conditions

$$\text{for } r = 0 \quad W = 0, \quad \text{for } r \to \infty \quad \frac{\partial W}{\partial r} \to 0, \quad \frac{1}{r}\frac{\partial W}{\partial \theta} \to 0;$$
$$\text{for } \theta = 0 \quad W = 0, \quad \text{for } \theta = \alpha \quad W = 0. \tag{7}$$

The solution of Eq. (6) is sought in series form:

$$W(r, \theta) = \sum_{n=1}^{\infty} W_n(r) \sin\frac{n\pi\theta}{\alpha}. \tag{8}$$

The coefficients of series $W_n(r)$ are determined from the equation

$$r^2 W_n'' + r W_n' - \left(\frac{n\pi}{\alpha}\right)^2 W_n = -\frac{2}{n\pi}\left(r^2 f'' + r f'\right). \tag{9}$$

Equation (9) is the Euler equation. It is integrated in elementary functions:

for region $r \le r_0$

$$W_n = C_1 r^{\frac{n\pi}{\alpha}} + C_2 r^{-\frac{n\pi}{\alpha}} - \frac{2/n\pi}{1 - \left(\frac{n\pi}{\alpha}\right)^2} k_1 r, \tag{10}$$

for region $r \ge r_0$

$$W_n = C_3 r^{\frac{n\pi}{\alpha}} + C_4 r^{-\frac{n\pi}{\alpha}} - \frac{1/2n\pi}{\frac{1}{4} - \left(\frac{n\pi}{\alpha}\right)^2} k_2 \sqrt{r}. \tag{11}$$

The constants of integration C_1, C_2, C_3, C_4 are obtained from the boundary conditions at the origin of the coordinates ($r = 0$) and at infinity ($r = \infty$) as well as from the condition for "splicing" the solutions at the line $r = r_0$. The resulting equation for the stream function of the flow under consideration has the form

$$\psi(r, \theta) = k_1 r \frac{\sin(\alpha - \theta)}{\sin\alpha} + \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}}\left[\frac{k_1 r_0 \alpha}{(n\pi)^2}\,\frac{1}{1 - \frac{n\pi}{\alpha}}\right.$$

$$-k_2 \frac{\alpha \sqrt{r_0}}{(2n\pi)^2} \frac{1}{\frac{1}{2}-\frac{n\pi}{\alpha}} - \frac{k_1 r_0 \alpha}{(n\pi)^2} + \frac{k_2 \alpha \sqrt{r_0}}{2(n\pi)^2}\Bigg] \sin \frac{n\pi\theta}{\alpha}, \tag{12}$$

for the region $r \leq r_0$;

$$\psi(r, \theta) = (k_1 r_0 - k_2 \sqrt{r_0}) \left(1 - \frac{\theta}{\alpha}\right) + k_2 \sqrt{r} \frac{\sin\left(\frac{\alpha}{2} - \frac{\theta}{2}\right)}{\sin \frac{\alpha}{2}}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{r_0}{r}\right)^{\frac{n\pi}{\alpha}} \left[\frac{k_1 r_0 \alpha}{(n\pi)^2} \frac{1}{1+\frac{n\pi}{\alpha}} - \frac{k_2 \alpha \sqrt{r_0}}{(2n\pi)^2} \frac{1}{\frac{1}{2}+\frac{n\pi}{\alpha}} - \frac{k_1 \alpha r_0}{(n\pi)^2} + \frac{k_2 \alpha \sqrt{r_0}}{2(n\pi)^2}\right] \sin \frac{n\pi\theta}{\alpha}, \tag{13}$$

for the region $r \geq r_0$.

The projection of the influx rate are defined by the derivatives of the stream function

$$v_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{\partial \psi}{\partial r}. \tag{14}$$

Considering the relations

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}} \frac{1}{n} \cos \frac{n\pi\theta}{\alpha} = -\frac{1}{2} \ln \left[1 - 2\left(\frac{r}{r_0}\right)^{\frac{\pi}{\alpha}} \cos \frac{\pi\theta}{\alpha} + \left(\frac{r}{r_0}\right)^{\frac{2\pi}{\alpha}}\right] \tag{15}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}} \frac{1}{n} \sin \frac{n\pi\theta}{\alpha} = \operatorname{arctg} \frac{\left(\frac{r}{r_0}\right)^{\frac{\pi}{\alpha}} \sin \frac{\pi\theta}{\alpha}}{1 - \left(\frac{r}{r_0}\right)^{\frac{\pi}{\alpha}} \cos \frac{\pi\theta}{\alpha}}, \tag{16}$$

the velocity components of the flow beyond the jet may be written as:

for the region $r \leq r_0$

$$v_r = k_1 \frac{\cos(\alpha - \theta)}{\sin \alpha} + \left(\frac{k_2 \sqrt{r_0}}{4\pi r} - \frac{k_1 r_0}{2\pi r}\right) \ln \left[1 - 2\left(\frac{r}{r_0}\right)^{\frac{\pi}{\alpha}} \cos \frac{\pi\theta}{\alpha} + \left(\frac{r}{r_0}\right)^{\frac{2\pi}{\alpha}}\right]$$

$$- \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}} \left[\frac{k_1 r_0}{\pi r} \frac{1}{n\left(1 - \frac{n\pi}{\alpha}\right)} - \frac{k_2 \sqrt{r_0}}{4\pi r} \frac{1}{n\left(\frac{1}{2} - \frac{n\pi}{\alpha}\right)}\right] \cos \frac{n\pi\theta}{\alpha}, \tag{17}$$

$$v_\theta = k_1 \frac{\sin(\alpha - \theta)}{\sin \alpha} + \left(\frac{k_2 \sqrt{r_0}}{2\pi r} - \frac{k_1 r_0}{\pi r}\right) \operatorname{arctg} \frac{\left(\frac{r}{r_0}\right)^{\frac{\pi}{\alpha}} \sin \frac{\pi\theta}{\alpha}}{1 - \left(\frac{r}{r_0}\right)^{\frac{\pi}{\alpha}} \cos \frac{\pi\theta}{\alpha}}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{\frac{n\pi}{\alpha}} \left[\frac{k_1 r_0}{\pi r} \frac{1}{n\left(1 - \frac{n\pi}{\alpha}\right)} - \frac{k_2 \sqrt{r_0}}{4\pi r} \frac{1}{n\left(\frac{1}{2} - \frac{n\pi}{\alpha}\right)}\right] \sin \frac{n\pi\theta}{\alpha}; \tag{18}$$

for the region $r \geq r_0$

$$v_r = \frac{k_1 r_0 - k_2 \sqrt{r_0}}{\alpha r} + \frac{k_2}{2\sqrt{r}} \frac{\cos \frac{\alpha - \theta}{2}}{\sin \frac{\alpha}{2}} + \left(\frac{k_2 \sqrt{r_0}}{4\pi r} - \frac{k_1 r_0}{2\pi r}\right) \ln \left[1 - 2\left(\frac{r_0}{r}\right)^{\frac{\pi}{\alpha}} \cos \frac{\pi\theta}{\alpha} + \left(\frac{r_0}{r}\right)^{\frac{2\pi}{\alpha}}\right]$$

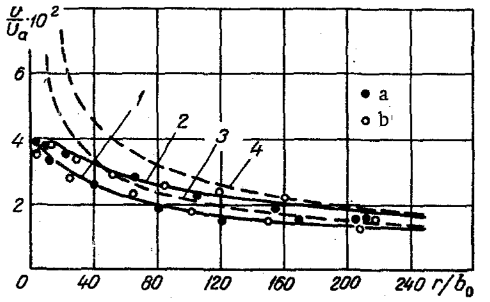

Fig. 3. Distribution of the influx rate along the directional beam $\theta = \alpha/2$. Calculated from (18)-(19): 1) $\beta = 3\pi/4$; 2) $\beta = \pi/2$; calculated from (21)-(23): 3) $\beta = 3\pi/4$; 4) $\beta = \pi/2$; experiment: a) $U_\alpha = 100$ m/sec; b) $U_\alpha = 150$ m/sec.

$$-\sum_{n=1}^{\infty}\left(\frac{r_0}{r}\right)^{\frac{n\pi}{\alpha}}\left[\frac{k_1 r_0}{\pi r}\,\frac{1}{n\left(1+\frac{n\pi}{\alpha}\right)}-\frac{k_2\sqrt{r_0}}{4\pi r}\,\frac{1}{n\left(\frac{1}{2}+\frac{n\pi}{\alpha}\right)}\right]\cos\frac{n\pi\theta}{\alpha}, \tag{19}$$

$$v_\theta=\frac{k_2}{2\sqrt{r}}\,\frac{\sin\frac{\alpha-\theta}{2}}{\sin\frac{\alpha}{2}}+\left(\frac{k_1 r_0}{\pi r}-\frac{k_2\sqrt{r_0}}{2\pi r}\right)\operatorname{arctg}\frac{\left(\frac{r_0}{r}\right)^{\frac{\pi}{\alpha}}\sin\frac{\pi\theta}{\alpha}}{1-\left(\frac{r_0}{r}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi\theta}{\alpha}}-\sum_{n=1}^{\infty}\left(\frac{r_0}{r}\right)^{\frac{n\pi}{\alpha}}\left[\frac{k_1 r_0}{\pi r}\right.$$

$$\left.\times\frac{1}{n\left(1+\frac{n\pi}{\alpha}\right)}-\frac{k_2\sqrt{r_0}}{4\pi r}\,\frac{1}{n\left(\frac{1}{2}+\frac{n\pi}{\alpha}\right)}\right]\sin\frac{n\pi\theta}{\alpha}. \tag{20}$$

For influx rates obtained from (17)-(20), the rarefaction created by the jet is determined from the Bernoulli integral. For $r_0 \to 0$, Eqs. (13), (19), (20) reduce to the known solution [2] for a fluid flow induced by a plane source-jet:

$$\psi(r,\theta)=k_2\sqrt{r}\,\frac{\sin\frac{\alpha-\theta}{2}}{\sin\frac{\alpha}{2}}, \tag{21}$$

$$v_r=\frac{k_2}{2\sqrt{r}}\,\frac{\cos\frac{\alpha-\theta}{2}}{\sin\frac{\alpha}{2}}, \tag{22}$$

$$v_\theta=\frac{k_2}{2\sqrt{r}}\,\frac{\sin\frac{\alpha-\theta}{2}}{\sin\frac{\alpha}{2}}. \tag{23}$$

Influx to the jet was studied using an arrangement that produced a plane air jet. The jet was expelled from nozzles with a half-width $2b_0$ of 28 and 10 mm. The exhaust velocities U_a at the nozzle exist section were 80, 100, and 150 m/sec. The velocity profile at the nozzle exist section was assumed to be uniform. The jet temperature was equal to the ambient temperature. The pressure distribution over the outer nozzle wall was measured with micromanometers, and the influx to the jet with a hot-wire anemometer. Figure 2 shows the distribution of the pressure coefficient $c_p = (p - p_\infty)/(\rho U_a^2/2)$ over the outer nozzle wall ($\theta = \alpha$) for two values of angle β, and Fig. 3 the distribution of the influx rate along the directional beam $\theta = \alpha/2$ for the same values of β. The figures also show the pressure and velocity profiles for the test conditions calculated both from the theory proposed (solid curves) and by the source-jet method (dashed curves). For an inclination angle $\beta = \pi/2$ of the nozzle outer wall with respect to the jet axis, pressure measurements have also been performed in [3]. The data obtained in [3] are in satisfactory agreement with our results (see Fig. 2). A certain quantitative discrepancy near the nozzle can be apparently interpreted by the difference in our and Wygnanski's experimental profiles.

It can be seen from (17)-(20) and from experiments that in dimensionless coordinates, the velocity and pressure profiles are universal with respect to the conditions at the nozzle exist section and that they depend on the angle formed by the jet axis and the nozzle wall. The comparison shows good agreement between the theoretical and experimental data, and reveals the importance of taking into account the inlet portion of the jet which has an appreciable effect on the nature of the influx at distances of up to 50 nozzle diameters.

NOTATION

r, θ	are polar system coordinates;
v_θ, v_r	are transverse and radial velocity components;
ψ	is the stream function;
U_a	is the exhaust velocity from the nozzle;
b_0	is the half-width of plane nozzle;
r_0	is the length of inlet portion of the jet;
α	is the angle between turbulent-jet boundary and nozzle wall;
β	is the angle between jet axis and nozzle wall;
ρ	is the density;
p	is the pressure;
p_∞	is the pressure in unperturbed medium.

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